

Some Remarks on the Hyperkähler Reduction

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We consider a *hyperkähler reduction* and describe it via frame bundles. Tracing the connection through the various reductions, we recover the results of [3]. In addition, we show that the fibers of such a reduction are necessarily totally geodesic. As an independent result, we describe O’Neill’s submersion tensors [6] on principal bundles.

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1 Introduction

The Hyperkähler Reduction is a cousin of the Symplectic Reduction applicable to the setting where the starting manifold M is hyperkähler and the involved data, the action of an auxiliary group G and the moment map μ , respect this structure. It is well known, that this implies that the *final manifold*, the quotient of a preimage of a central regular value of μ by G , also is a hyperkähler manifold. This however is not all that is special about the hyperkähler reduction.

In their paper [3] T. Gocho and H. Nakajima find some interesting relations between various geometrical quantities involved in this construction. The paper uses various calculations in the tangent bundle to show these relations.

We will present a different approach in this work by *lifting* the calculation onto the involved principal bundles. Although quite a bit longer than the original work, it highlights the role the quaternionic structure plays in the construction. The length can be partly attributed to the need to introduce basic notions in this setting, e.g. the section 4.5 *Riemannian Submersions* which recovers the fundamentals of O'Neill's theory in the principal bundle setting.

The aim of this paper is to show that these relations can be derived fundamentally from the structure of quaternionic matrices, when embedded into real matrices. It does so, by first deriving equation (68), which does not need the involved quaternionic structures. Then this equation is compared to the *quaternionic world* (69), and this comparison yields all the relations that we long for. It then just remains to decipher the implied relations for the quaternionic components.

The section 2 *Definitions* recalls the basic notions involved in hyperkähler

geometry and in particular in a hyperkähler reduction. Of utmost importance to the next sections are the notions of reduction and extension of principal bundles. Further it describes a recipe to compare forms on the manifolds and the involved principal bundles.

Section 3 *Setting* first discusses the tangent bundle of M and how its quaternionic structure behaves with respect to the reduction. This structure allows for various reductions of the principal bundle of frames of M . These bundles lie at the heart of the construction in this work.

The following section inspects the involved forms with respect to the bundles discussed. Concretely we will trace the reductions of the Levi-Civita connection and tautological form starting from the principal bundle of frames of M all the way to the principal bundle of frames of the quotient N . A quick excursion is made in this section, explaining the fundamentals of Riemannian Submersions in the principal bundle language.

The last section 5 *Final Results* uses the preceding work to recover the results of Gocho and Nakajima, and show a small novelty. It is this section where the relation between the quaternionic structure and the results is investigated.

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2 Definitions

Let us define some standard notions. Throughout this paper, let M be a smooth oriented Riemannian manifold of dimension $4m \in \mathbb{N}$, and G a smooth Lie group of dimension $k \in \mathbb{N}$.

Notation 2.1. By $\text{Fr}_{\text{SO}}(M)$ we denote the *principal bundle of orthonormal frames* on M ,

$$\text{Fr}_{\text{SO}}(M) = \{p: \mathbb{R}^{4m} \rightarrow T_x M : p \text{ is an oriented orthogonal isomorphism}\}.$$

Notation 2.2. By $\theta^M \in \Omega^1(\text{Fr}_{\text{SO}}(M), \mathbb{R}^{4m})$ we denote the *soldering form* of $\text{Fr}_{\text{SO}}(M)$

$$\theta_p^M(\xi) = p^{-1} \circ D\pi_p(\xi), \quad p \in \text{Fr}_{\text{SO}}(M), \quad \xi \in T_p \text{Fr}_{\text{SO}}(M),$$

where $\pi: \text{Fr}_{\text{SO}}(M) \rightarrow M$ is the projection.

Let $\varphi \in \Omega^1(\text{Fr}_{\text{SO}}(M), \mathfrak{so}(4m))^{\text{SO}(4m)}$ denote the Levi-Civita connection of (M, g) . Then φ satisfies

- $R_g^* \varphi = \text{Ad}_{g^{-1}} \circ \varphi$, for all $g \in \mathbf{SO}(4m)$,
- $\varphi(K^\xi) = \xi$ for all $\xi \in \mathfrak{so}(4m)$, where K^ξ is the fundamental vector field to the lie algebra element ξ , i.e.

$$K_p^\xi = \left. \frac{d}{dt} \right|_{t=0} (p \exp(t\xi)),$$

- $d\theta + \varphi \wedge \theta = 0$, i.e. φ has zero torsion.

Definition 2.3 (Hyperkähler Manifold). A Riemannian manifold (M, g) with a triple of almost complex structures I, J, K ,

$$I, J, K: TM \rightarrow TM, \quad I^2 = J^2 = K^2 = -\text{id}_{TM},$$

which satisfy the quaternionic relation $IJ = K$ and are compatible with the metric,

$$g(-, -) = g(I-, I-) = g(J-, J-) = g(K-, K-),$$

is called a hyperkähler manifold (hk-manifold) if the two-forms corresponding to I, J and K are closed, i.e.

$$d\omega_A = 0, \quad \omega_A(-, -) = g(A-, -), \quad A \in \{I, J, K\}.$$

Proposition 2.4 (Alternative Characterization). (M^{4m}, g) is a hyperkähler manifold if and only if the structure group of $\text{Fr}_{\mathbf{SO}}(M)$ reduces to $\mathbf{Sp}(m)$ and the Levi-Civita connection on $\text{Fr}_{\mathbf{SO}}(M)$ reduces to a connection on

$$\text{Fr}_{\mathbf{Sp}}(M) = \{p: \mathbb{H}^m \rightarrow T_x M : p \text{ is a } \mathbb{H}\text{-linear isomorphism}\},$$

i.e. the horizontal subspaces are tangent to the submanifold $\text{Fr}_{\mathbf{Sp}}(M) \subset \text{Fr}_{\mathbf{SO}}(M)$.

Note that in the dual formulation the condition on the horizontal subspaces is that φ reduces to a connection on $\text{Fr}_{\mathbf{Sp}}(M)$. Precisely this means that $\lambda_* j^* \varphi$ is a connection on $\text{Fr}_{\mathbf{Sp}}(M)$, where $j: \text{Fr}_{\mathbf{Sp}}(M) \rightarrow \text{Fr}_{\mathbf{SO}}(M)$ and $\lambda: \mathbf{Sp}(m) \rightarrow \mathbf{SO}(4m)$ are the inclusions and $\lambda_*: \mathfrak{sp}(m) \rightarrow \mathfrak{so}(4m)$ is the derivative of λ .

Definition 2.5 (Hyperkähler Action). We say a group G acts hyperkähler on a hyperkähler manifold (M, g, I, J, K) , if G acts on M and this action preserves the metric g and the hyperkähler structures I, J and K , i.e.

$$R_h^* \omega_A = \omega_A \quad \forall A \in \{I, J, K\}, \quad R_h^* g = g, \quad (1)$$

for all $h \in G$. (In this case we used a right action of G on M , but this definition does not require so).

Definition 2.6 (tri-hamiltonian action). A hyperkähler action of G on M is called a tri-hamiltonian action, if G -equivariant moment maps

$$\mu_I, \mu_J, \mu_K: M \rightarrow \mathfrak{g}^* \quad (2)$$

exist, i.e.

$$\mu_A(x.h) = \text{Ad}_{h^{-1}}^* \circ \mu_A(x) \quad \forall x \in M, \quad \forall h \in G, \quad \forall A \in \{I, J, K\}, \quad (3)$$

$$\langle \xi, d\mu_A(\eta) \rangle = \omega_A(K^\xi, \eta) \quad \forall \eta \in TM, \quad \forall \xi \in \mathfrak{g}, \quad \forall A \in \{I, J, K\}. \quad (4)$$

The moment maps of a tri-hamiltonian action are also often considered together as a map $\mu = (\mu_I, \mu_J, \mu_K): M \rightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*$.

2.1 Reduction and Extensions

Let $\pi: P \rightarrow M$ be a principal bundle with structure group G . A reduction of P is a principal bundle $Q \rightarrow M$ with structure group H and maps

$$\lambda: H \rightarrow G, \quad f: Q \rightarrow P, \quad (5)$$

a Lie homomorphism and a smooth map respectively, such that the following diagram commutes.

$$\begin{array}{ccc} Q \times H & \xrightarrow{f \times \lambda} & P \times G \\ \downarrow & & \downarrow \\ Q & \xrightarrow{f} & P \\ & \searrow & \swarrow \\ & M & \end{array}$$

The vertical maps above are the group actions on the principal bundles. An extension of P is a principal bundle $\tilde{Q} \rightarrow M$ of structure group \tilde{H} with maps $\tilde{\lambda}: G \rightarrow \tilde{H}$ and $\tilde{f}: P \rightarrow \tilde{Q}$, such that P is a reduction of \tilde{Q} .

Given a connection ϕ^P on P , then there is a unique connection $\phi^{\tilde{Q}}$ on \tilde{Q} such that

$$\tilde{f}^* \phi^{\tilde{Q}} = \tilde{\lambda}_* \circ \phi^P, \quad (6)$$

where $\tilde{\lambda}_*$ is the derivative of $\tilde{\lambda}$ (see e.g. [1, Satz 4.1]). In this sense, a connection is always extendable. If two connections satisfy the equation above, we say that ϕ^P extends to $\phi^{\tilde{Q}}$ and $\phi^{\tilde{Q}}$ reduces to ϕ^P .

On Q the situation is somewhat more complicated. We will only discuss the situation for the simplest case where $f = i$ and λ are the inclusions.

Proposition 2.7 (Reduction of a connection). *If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{f}$ as H -representations, i.e. $\mathfrak{f} \subset \mathfrak{g}$ is a vector space complement of $\mathfrak{h} \subset \mathfrak{g}$, with the property that*

$$\text{Ad}_H(\mathfrak{f}) \subset \mathfrak{f}, \quad (7)$$

then $\text{pr}_{\mathfrak{h}} \circ i^ \phi^P$ is a connection on Q , where the projection is with respect to the decomposition given above.*

Proof. The only thing to note is, that the condition $\text{Ad}_H(\mathfrak{f}) \subset \mathfrak{f}$ (together with $\text{Ad}_H(\mathfrak{h}) \subset \mathfrak{h}$) implies that $\text{pr}_{\mathfrak{h}}$ commutes with Ad_h for all $h \in H$. The necessary conditions are then easily checked. \square

Definition 2.8. We say that ϕ *reduces* to Q when the horizontal subspaces are tangent to the subbundle $Q \subset P$. In the dual formulation this is true if and only if the pulled back connection takes values in the Lie algebra \mathfrak{h} , so that no projection is necessary.

Note that a projected connection as in the lemma above can be extended back to P . This will however yield a different connection if the original one was not reducible. This also implies that there are in general multiple connections on P that project onto a given connection on Q .

Remark 2.9. Let $\iota: Q \rightarrow \text{Fr}_{\text{SO}}(M)$ denote a reduction of the frame bundle. We call the pull back θ^Q of θ^M to Q again *soldering form* of Q . Since the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & \text{Fr}_{\text{SO}}(M) \\ & \searrow \pi^Q & \swarrow \pi^M \\ & M & \end{array}$$

commutes, we have that for all $p \in Q$ and $\xi \in T_p Q$

$$\theta_p^Q(\xi) = (\iota^* \theta^M)_p(\xi) = \theta_{\iota(p)}^M(\xi) = \iota(p)^{-1} \circ (D\pi^M)_{\iota(p)} \circ D\iota_p(\xi) \quad (8)$$

$$= \iota(p)^{-1} \circ (D(\pi^M \circ \iota))_p(\xi) = \iota(p)^{-1} \circ (D\pi^Q)_p(\xi), \quad (9)$$

so that $\theta_p^Q = \iota(p)^{-1} \circ D\pi_p^Q$. In this sense the construction is natural.

2.2 The Correspondence of Forms

Having a principal bundle of frames $\text{Fr}_{\mathbf{GI}}(M)$ (or any reduction of it) over a manifold M induces a correspondence between certain forms on the base manifold and the bundle. We will use this correspondence to compare our approach and the one taken in [3].

Lemma 2.10 (Correspondence of forms). *There is a one-to-one correspondence between horizontal, equivariant and $\mathfrak{gl}(4m)$ -valued one-forms on the principal bundle of frames, and (global) sections of the vector bundle $T^*M \otimes \text{End}(TM)$.*

Remark 2.11. Note that this is a special case of the correspondence between representation valued forms on a principal bundle and forms with values in associated vector bundles on the base. In the presence of the soldering form, we can give a simple explicit description.

Proof. Let ω be a horizontal and equivariant one-form on the principal bundle. We induce the wanted section as follows. If $x \in M$ and $\xi, \eta \in T_x M$, let $p \in \text{Fr}_{\mathbf{GI}}(M)$ be any frame in the fiber of π over x . Define

$$s(\omega)(\xi, \eta) = p\omega(\tilde{\xi})\theta(\tilde{\eta}), \quad (10)$$

where θ is the solder form of $\text{Fr}_{\mathbf{GI}}(M)$ and $\tilde{\xi}$ and $\tilde{\eta}$ are lifts of ξ and η to $p \in \text{Fr}_{\mathbf{GI}}(M)$, i.e. $D\pi(\tilde{\xi}) = \xi$ and $D\pi(\tilde{\eta}) = \eta$. This is well defined, because for a different choice of lifts $\tilde{\xi}$ and $\tilde{\eta}$, the differences $\Delta\xi = \tilde{\xi} - \tilde{\xi}$ and $\Delta\eta = \tilde{\eta} - \tilde{\eta}$ are vertical, but ω and θ are both horizontal forms. A different choice of frame $q = p.g \in \text{Fr}_{\mathbf{GI}}(M)$, leads to the calculation

$$\begin{aligned} q\omega(\tilde{\xi})\theta(\tilde{\eta}) &= q\omega(\tilde{\xi})q^{-1}(\eta) = p.g\omega(\tilde{\xi})(p.g)^{-1}(\eta) = pg\omega(\tilde{\xi})g^{-1}p^{-1}(\eta) \\ &= p\text{Ad}_g(\omega(\tilde{\xi}))\theta(\tilde{\eta}) = p\omega(DR_{g^{-1}}\tilde{\xi})\theta(\tilde{\eta}) = p\omega(\tilde{\xi})\theta(\tilde{\eta}) \\ &= p\omega(\tilde{\xi})\theta(\tilde{\eta}), \end{aligned} \quad (11)$$

where we have used the equivariance of ω , $R_g^*\omega = \text{Ad}_{g^{-1}}\omega$, and the fact that DR_g maps lifts into lifts, since $\pi \circ R_g = \pi$ and therefore $D\pi \circ DR_g = D\pi$ for all $g \in \mathbf{GI}(m)$. By abuse of notation $\tilde{\eta}$ denotes a lift to both q and p in $T\text{Fr}_{\mathbf{GI}}(m)$.

Note that we have only needed $\mathbf{GI}(m)$ for the fact that $\text{Ad}_g(\xi) = g\xi g^{-1}$, so this will be true for all principal bundles in this work, if we adjust the vector bundle in which the sections are taken.

The inverse map, sending a section to a form on the principal bundle is defined by

$$\omega(s)(\xi) = p^{-1}s(D\pi(\xi))p, \quad (12)$$

where $p \in \text{Fr}_{\mathbf{GL}}(M)$ is some frame, $\xi \in T_p \text{Fr}_{\mathbf{GL}}(M)$ and $s \in \Gamma(T^*M \otimes \text{End}(TM))$ is the section. This form is clearly a horizontal $\mathfrak{gl}(\mathfrak{m})$ -valued one-form. It is also equivariant because

$$\begin{aligned} R_g^* \omega(s)(\xi) &= (pg)^{-1} s(D\pi \circ DR_g(\xi)) pg = g^{-1} p^{-1} s(D\pi(\xi)) pg \\ &= \text{Ad}_{g^{-1}} \omega(s)(\xi). \end{aligned} \quad (13)$$

It is easy to show that these two maps are inverse of each other, which concludes the proof. \square

Definition 2.12 (Corresponding forms). As denoted in the proof above, the section of $T^*M \otimes \text{End}(TM)$ corresponding to ω is denoted by $s(\omega)$, and the form corresponding to a section s by $\omega(s)$.

Note that this result remains true for reductions of the basis bundle, if we adjust the vector bundle in which the sections are taken. For example, the above mentioned forms on $\text{Fr}_{\mathbf{SO}}(M)$ correspond to sections in $T^*M \otimes \mathfrak{so}(TM)$ and the forms on $\text{Fr}_{\mathbf{Sp}}(M)$ to sections of $T^*M \otimes \mathfrak{sp}(TM)$.

Example 2.13 (Difference form). A well known example of this correspondence is between the difference form of two connections on a principal bundle, and the difference tensor of the two associated covariant derivatives. This follows immediately from equation (38).

3 Setting

We will recover the results from [3] for principal bundles.

Let (M, g) be an Riemannian manifold of dimension $4m \in \mathbb{N}$, and let $M \leftarrow G$ be a tri-hamiltonian action of G on M . Let $k \in \mathbb{N}$ be the dimension of the Lie group G . We denote the momentum map by $\mu: M \rightarrow \mathbb{R}^3 \otimes \mathfrak{g}^*$. We assume that $0 \in \mathbb{R}^3 \otimes \mathfrak{g}^*$ is a regular value of μ . This implies that G acts on the submanifold $\mu^{-1}(0)$, because equation (3) guarantees that for $x \in \mu^{-1}(0)$, i.e. $\mu_A(x) = 0$ for all A , we have

$$\mu_A(x.h) = \text{Ad}_h^* \circ \mu_A(x) = 0, \quad \forall h \in G, \quad (14)$$

and hence $x.h \in \mu^{-1}(0)$.

We assume further that this action is free and proper, so that the quotient $\mu^{-1}(0)/G$ is a Hausdorff space, and define $N := \mu^{-1}(0)/G$.

$$\begin{array}{ccc}
\mu^{-1}(0) & \xrightarrow{\iota} & M \\
\downarrow \pi & & \\
N & &
\end{array}$$

We will show that N also is a hyperkähler manifold, and that the second fundamental form of $\mu^{-1}(0)$ in M is given by the Hessian of μ , compare [3] and [4].

3.1 The Splitting of TM

The tri-hamiltonian action $M \hookleftarrow G$ splits the vector bundle TM over $\mu^{-1}(0)$, i.e. the ambient bundle

$$\iota^*TM, \tag{15}$$

in the following way.

Proposition 3.1. *If $x \in \mu^{-1}(0)$, we have*

$$T_x M = T_x \mu^{-1}(0) \oplus T_x \mu^{-1}(0)^\perp = H_x \oplus \mathfrak{g} \oplus T_x \mu^{-1}(0)^\perp, \tag{16}$$

where $\mathfrak{g} \subset T_x M$ is defined by the fundamental vector fields, i.e. the image of $K: \mathfrak{g} \rightarrow \Gamma(TM)$, and H_x is the orthogonal complement to \mathfrak{g} in $T_x \mu^{-1}(0)$ with respect to the metric g . All direct sums are orthogonal.

Then H_x is a quaternionic subspace of $T_x M$ and

$$T_x \mu^{-1}(0)^\perp = I\mathfrak{g} \oplus J\mathfrak{g} \oplus K\mathfrak{g}. \tag{17}$$

Proof. If $\xi \in \mathfrak{g}$ and $\eta \in T_x \mu^{-1}(0)$ then η is tangent to a level set of μ , i.e. $d\mu(\eta) = 0$, which implies for $A \in \{I, J, K\}$

$$g(AK^\xi, \eta) = \omega_A(K^\xi, \eta) = \langle \xi, d\mu_A(\eta) \rangle = 0, \tag{18}$$

hence $AK^\xi \in T_x \mu^{-1}(0)^\perp$ for all A .

Furthermore the sets $I\mathfrak{g}$, $J\mathfrak{g}$ and $K\mathfrak{g}$ have a trivial intersection. Indeed, assume $\xi, \eta \in \mathfrak{g}$ with $I\xi = J\eta$. Then $K\xi = \eta$ but since $K\xi$ is in $T_x \mu^{-1}(0)^\perp$, $\eta = \xi = 0$.

Since the codimension of $\mu^{-1}(0)$ in M is $3k$, where $k = \dim G = \dim \mathfrak{g}$, we see that

$$T_x \mu^{-1}(0)^\perp = I\mathfrak{g} \oplus J\mathfrak{g} \oplus K\mathfrak{g}. \tag{19}$$

Finally, I, J and K let the orthogonal complement of H_x invariant and are orthogonal, so they also let H_x invariant. \square

We conclude that TM splits over $\mu^{-1}(0)$ into two quaternionic sub-bundles

$$TM = H \oplus \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}. \quad (20)$$

Notice that while the first bundle has a quaternionic structure, the second one has a quaternionic and a real structure. This will become important later on.

The metric g of M induces a metric on H . Since $M \hookrightarrow G$ is hyperkähler and g is G -invariant it furnishes N with a Riemannian metric. Similarly the quaternionic structure on M induces one on H (because of the quaternionic decomposition above), which in turn induces one on N compatible with the metric. This reduces the principal bundle of orthogonal frames on N to the structure group $\mathbf{Sp}(n)$ ($n = m - k$, $4n$ is the dimension of N). We will show later that the connection of N reduces so that N is indeed a hyperkähler manifold.

3.2 The Principal Bundles

Similar to the vector bundle TM , we may depict the splitting in the principal bundle setting. Fix a splitting

$$\mathbb{R}^{4m} = \mathbb{R}^{4n} \oplus \mathbb{R}^k \oplus \mathbb{R}^{3k} = \mathbb{H}^n \oplus \mathbb{H}^k \quad (21)$$

Now we can ask frames $p: \mathbb{R}^{4m} \rightarrow T_x M$ to respect various degrees of the structure. Let $x \in \mu^{-1}(0)$.

- $p: \mathbb{R}^{4m} \rightarrow T_x M$ with no condition at all. These frames are in the pull back of the frame bundle $\text{Fr}_{\mathbf{SO}}(M)$ to $\mu^{-1}(0)$, denoted by $\iota^* \text{Fr}_{\mathbf{SO}}(M)$.
- $p: \mathbb{R}^{4m} \rightarrow T_x M$ with $p(\mathbb{R}^{4n} \oplus \mathbb{R}^k) = T_{\mu^{-1}(0)}$, frames adapted to the submanifold $\mu^{-1}(0) \subset M$. This is a principal bundle whose structure group is $\mathbf{SO}(4n + k) \times \mathbf{SO}(3k)$, corresponding to the possible rotations of the frame in $T_{\mu^{-1}(0)}$ and $T_{\mu^{-1}(0)}^{\perp}$. We denote it by

$$\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), M) = \left\{ p \in \text{Fr}_{\mathbf{SO}}(M) : \text{im}(p|_{\mathbb{R}^{4n+k}}) = T_{\mu^{-1}(0)} \right\}, \quad (22)$$

- $p: \mathbb{R}^{4n} \oplus \mathbb{R}^k \rightarrow T_x \mu^{-1}(0)$. These frames can be identified with frames of $\mu^{-1}(0)$. We denote them with $\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0))$.

- $p: \mathbb{R}^{4n} \oplus \mathbb{R}^k \rightarrow T_x \mu^{-1}(0)$ with $p(\mathbb{R}^{4n}) = H_x$. These frames are frames of $\mu^{-1}(0)$ adapted to the fibration $\pi: \mu^{-1}(0) \rightarrow N$. The principal bundle of these have structure group $\mathbf{SO}(4n) \times \mathbf{SO}(k)$ corresponding to the rotations in the fiber and its orthogonal complement. We denote the bundle by

$$\mathrm{Fr}_{\mathbf{SO}}(N, \mu^{-1}(0)) = \left\{ p \in \mathrm{Fr}_{\mathbf{SO}}(\mu^{-1}(0)) : \quad \mathrm{im}(p|_{\mathbb{R}^{4n}}) = H_x \right\}, \quad (23)$$

- $p: \mathbb{R}^{4n} \rightarrow H_x$. The principal bundle of these frames can be identified with the pull back of $\mathrm{Fr}_{\mathbf{SO}}(N)$ to $\mu^{-1}(0)$ (note that we know already that N is a Riemannian manifold). We denote it by $\pi^* \mathrm{Fr}_{\mathbf{SO}}(N)$.

We may restrict the principal bundles above to quaternionic frames where it makes sense. Fix

$$\mathbb{H}^m = \mathbb{H}^n \oplus \mathbb{H}^k \quad (24)$$

respecting (21). This induces the following bundles, where all frames are \mathbb{H} -linear.

- $p: \mathbb{H}^m \rightarrow T_x M$ are the frames that make up the pull back of

$$\mathrm{Fr}_{\mathbf{Sp}}(M) = \{ p \in \mathrm{Fr}_{\mathbf{SO}}(M) : p \text{ is } \mathbb{H}\text{-linear} \} \quad (25)$$

to $\mu^{-1}(0)$. It is naturally a reduction of $\iota^* \mathrm{Fr}_{\mathbf{SO}}(M)$ to quaternionic frames, has structure group $\mathbf{Sp}(m)$ and will be denoted by $\iota^* \mathrm{Fr}_{\mathbf{Sp}}(M)$.

- $p: \mathbb{H}^m \rightarrow T_x M$ with $p(\mathbb{H}^n) = H_x$ and $p(\mathbb{H}^k) = \mathfrak{g} \otimes \mathbb{H}$ respecting both the quaternionic and real structure. We denote this principal bundle with structure group $\mathbf{Sp}(n) \times \mathbf{SO}(k)$ by

$$\mathrm{Fr}_{\mathbf{Sp}}(N, M) = \left\{ p \in \iota^* \mathrm{Fr}_{\mathbf{Sp}}(M) : \quad \mathrm{im}(p|_{\mathbb{H}^n}) = H_x, \quad \mathrm{im}(p|_{\mathbb{H}^k}) = \mathfrak{g} \otimes \mathbb{H} \right\} \quad (26)$$

The frames are adapted to the quaternionic splitting of $T_x M = H_x \oplus \mathfrak{g} \otimes \mathbb{H}$ and respect the real structure of the second, $p(\mathrm{Re}(\mathbb{H}^k)) = \mathrm{Re}(\mathfrak{g} \otimes \mathbb{H}) = \mathfrak{g}$, so in particular (because I, J, K are orthogonal) respect the splitting $T_{\mu^{-1}(0)} \oplus T_{\mu^{-1}(0)}^\perp$.

- $p: \mathbb{H}^n \rightarrow H_x$ are the frames of the pulled back bundle $\mathrm{Fr}_{\mathbf{Sp}}(N)$ to $\mu^{-1}(0)$ and is denoted by $\pi^* \mathrm{Fr}_{\mathbf{Sp}}(N)$.

There are plenty of natural maps between these bundles. We will be using the following.

- *Reductions to quaternionic frames, denoted by i :* Some of the real frame bundles can be reduced to quaternionic frames, which induces maps from the quaternionic world to the real world. This is obviously the case for $\text{Fr}_{\mathbf{Sp}}(M) \rightarrow \text{Fr}_{\mathbf{SO}}(M)$, $\text{Fr}_{\mathbf{Sp}}(N) \rightarrow \text{Fr}_{\mathbf{SO}}(N)$ and their pull backs to $\mu^{-1}(0)$. Finally this is also the case for $\text{Fr}_{\mathbf{Sp}}(N, M) \rightarrow \text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), M)$, because a quaternionic frame that respects the splitting $T_x \mu^{-1}(0) \oplus T_x \mu^{-1}(0)^\perp$, automatically respects the quaternionic splitting $H_x \oplus \mathfrak{g} \otimes \mathbb{H}$, as can be seen by applying one of the complex structures to $T_x \mu^{-1}(0)^\perp$. In other words, $(\mathbf{SO}(4n+k) \times \mathbf{SO}(3k)) \cap \mathbf{Sp}(m) \cong \mathbf{Sp}(n) \times \mathbf{SO}(k)$.
- *Reduction to more structured frames, denoted by j :* Some of the bundles are simply restrictions of other bundles to frames respecting more structures. This is the case for

$$\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), M) \rightarrow \iota^* \text{Fr}_{\mathbf{SO}}(M), \quad \text{Fr}_{\mathbf{SO}}(N, \mu^{-1}(0)) \rightarrow \text{Fr}_{\mathbf{SO}}(\mu^{-1}(0)) \quad (27)$$

and

$$\text{Fr}_{\mathbf{Sp}}(N, M) \rightarrow \iota^* \text{Fr}_{\mathbf{Sp}}(M). \quad (28)$$

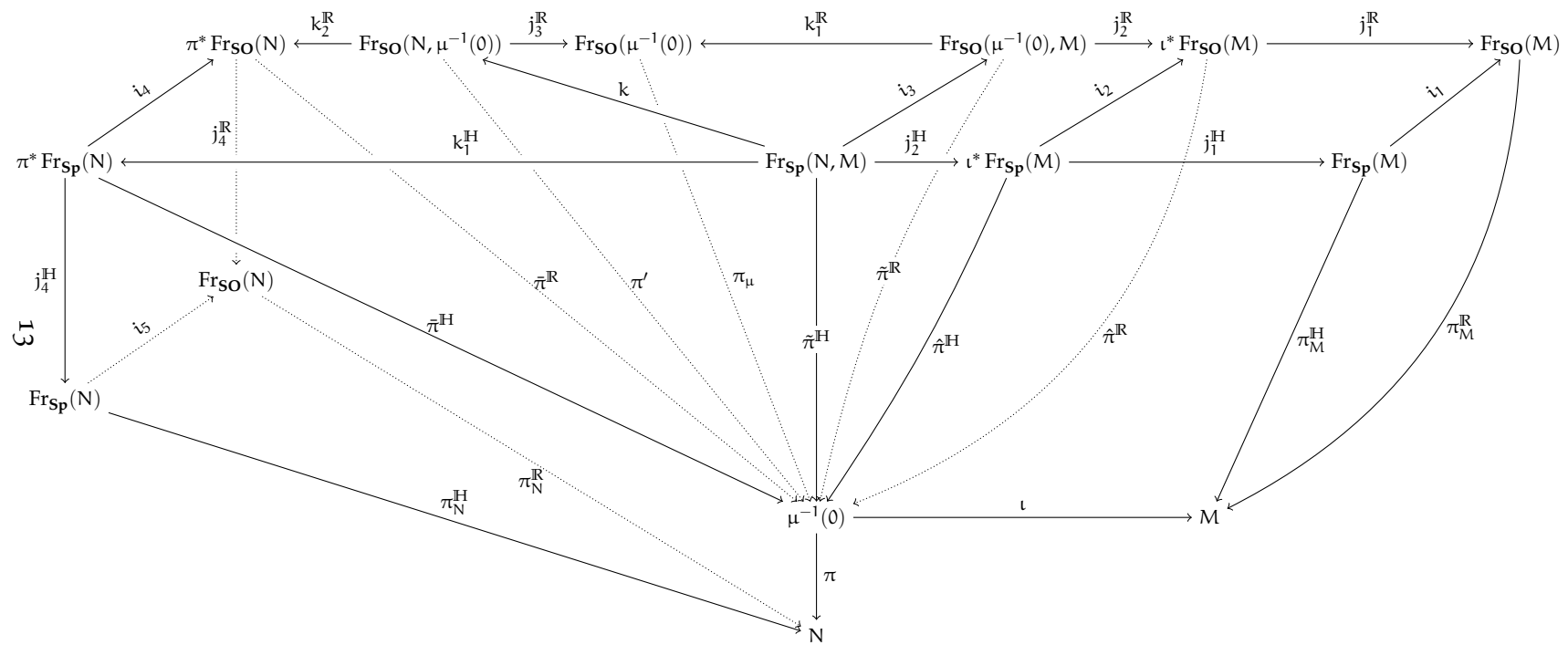
- *Induced maps by pull backs, also denoted by j :* There are of course canonical maps $\iota^* \text{Fr}_{\mathbf{SO}}(M) \rightarrow \text{Fr}_{\mathbf{SO}}(M)$ and similar for $\pi: \mu^{-1}(0) \rightarrow N$ and the quaternionic bundles.
- *Restrictions of frames, denoted by k :* Some bundles allow natural projections to other bundles by restricting the frame to a subspace of its domain. This is the case for

$$\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), M) \rightarrow \text{Fr}_{\mathbf{SO}}(\mu^{-1}(0)), \quad \text{Fr}_{\mathbf{SO}}(N, \mu^{-1}(0)) \rightarrow \pi^* \text{Fr}_{\mathbf{SO}}(N) \quad (29)$$

and

$$\text{Fr}_{\mathbf{Sp}}(N, M) \rightarrow \pi^* \text{Fr}_{\mathbf{Sp}}(N), \quad \text{Fr}_{\mathbf{Sp}}(N, M) \rightarrow \text{Fr}_{\mathbf{SO}}(N, \mu^{-1}(0)). \quad (30)$$

The aforementioned bundles are depicted in the following diagram.



4 The Induced Connections

In this chapter we will start with the Levi Civita connection on $\text{Fr}_{\text{SO}}(M)$ and chase it through the diagram. This will show that N is indeed a hyperkähler manifold and recover the results from [3].

4.1 Forms on $\text{Fr}_{\text{Sp}}(M)$

Starting with the solder form $\theta^{M,\mathbb{R}}$ and the Levi Civita connection $\varphi^{M,\mathbb{R}}$ on $\text{Fr}_{\text{SO}}(M)$, we first induce the forms $\theta^{M,\mathbb{H}}$ and $\varphi^{M,\mathbb{H}}$ on $\text{Fr}_{\text{Sp}}(M)$, by pulling back with i_1 ,

$$\theta^{M,\mathbb{H}} = i_1^* \theta^{M,\mathbb{R}}, \quad \varphi^{M,\mathbb{H}} = i_1^* \varphi^{M,\mathbb{R}}.$$

Since M is a hk-manifold, $\varphi^{M,\mathbb{H}}$ is a connection on $\text{Fr}_{\text{Sp}}(M)$ satisfying the pulled back structure equation

$$d\theta^{M,\mathbb{H}} + \varphi^{M,\mathbb{H}} \wedge \theta^{M,\mathbb{H}} = 0.$$

As remarked in (2.9) $\theta^{M,\mathbb{H}}$ is again the soldering form of $\text{Fr}_{\text{Sp}}(M)$, hence $\varphi^{M,\mathbb{H}}$ is a torsion free connection on $\text{Fr}_{\text{Sp}}(M)$.

4.2 Forms on $\iota^* \text{Fr}_{\text{SO}}(M)$ and $\iota^* \text{Fr}_{\text{Sp}}(M)$

The solder forms and connection forms on $\text{Fr}_{\text{SO}}(M)$ and $\text{Fr}_{\text{Sp}}(M)$ further induce connections on the ambient principal bundles $\iota^* \text{Fr}_{\text{Sp}}(M)$ and $\iota^* \text{Fr}_{\text{SO}}(M)$ which we will denote by $\hat{\varphi}^{\mathbb{R}}, \hat{\theta}^{\mathbb{R}}$ and $\hat{\varphi}^{\mathbb{H}}, \hat{\theta}^{\mathbb{H}}$ with the obvious choice. The $\hat{\varphi}$ are connections, since we do not change the fibers of the principal bundle (although some may be discarded). It is also a torsion free connection, since the structural equation $d\theta + \varphi \wedge \theta = 0$ survives the pull back and by using remark (2.9), the pulled back solder forms are natural

$$\hat{\theta}_p^{\mathbb{R}}(\xi) = p^{-1} \circ D\hat{\pi}_p^{\mathbb{R}}(\xi), \quad \hat{\theta}_q^{\mathbb{H}}(\eta) = q^{-1} \circ D\hat{\pi}_q^{\mathbb{H}}(\eta),$$

where $p \in \iota^* \text{Fr}_{\text{SO}}(M)$, $\xi \in T_p \iota^* \text{Fr}_{\text{SO}}(M)$ and $q \in \iota^* \text{Fr}_{\text{Sp}}(M)$, $\eta \in T_q \iota^* \text{Fr}_{\text{Sp}}(M)$.

4.3 Forms on $\text{Fr}_{\text{SO}}(\mu^{-1}(0), M)$

The next step is to transfer these forms to the principal bundle

$$\text{Fr}_{\text{SO}}(\mu^{-1}(0), M) = \left\{ p \in \iota^* \text{Fr}_{\text{SO}}(M) : \text{im}(p|_{\mathbb{R}^{4n+k}}) = T\mu^{-1}(0) \right\},$$

which has structure group $\mathbf{SO}(4n+k) \times \mathbf{SO}(3k)$.

Different to before is that $\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), M)$ is in general not horizontal in the ambient bundle, hence we need to project in order to get a connection.

Lemma (2.7) allows us to define connections on the adapted frame bundles $\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), M)$ and $\text{Fr}_{\mathbf{Sp}}(N, M)$. With the inclusion

$$i: \mathbf{SO}(4n+k) \times \mathbf{SO}(3k) \rightarrow \mathbf{SO}(4m), \quad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

we get the Lie algebra decomposition (as vector spaces)

$$\mathfrak{so}(4m) = \mathfrak{so}(4n+k) \oplus \mathfrak{so}(3k) \oplus \mathfrak{f},$$

where

$$\mathfrak{f} = \left\{ \begin{pmatrix} 0 & C \\ -C^t & 0 \end{pmatrix} \in \mathfrak{so}(4m) : C \in \text{Mat}(4n+k, 3k) \right\}.$$

If $A \in \text{im}(i)$ and $\xi \in \mathfrak{f}$, then $\text{Ad}_A(\xi) = A\xi A^{-1} \in \mathfrak{f}$, hence we have a connection $\tilde{\varphi}^{\mathbb{R}} = \text{pr}_{\mathfrak{so}(4n+k) \oplus \mathfrak{so}(3k)} \circ j_2^{\mathbb{R}*} \hat{\varphi}^{\mathbb{R}}$ on $\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), M)$. This connection naturally decomposes into two equivariant one-forms $\phi_1^{\mathbb{R}}$ and $\phi_2^{\mathbb{R}}$ with values in $\mathfrak{so}(4n+k)$ and $\mathfrak{so}(3k)$ respectively.

We can go ahead and extend $\tilde{\varphi}^{\mathbb{R}}$ back to $\iota^* \text{Fr}_{\mathbf{SO}}(M)$, which gives us a connection $\hat{\varphi}^{\mathbb{R}}$. The difference form

$$\hat{\tau}^{\mathbb{R}} = \hat{\varphi}^{\mathbb{R}} - \tilde{\varphi}^{\mathbb{R}}, \quad (31)$$

is a equivariant horizontal one form, hence the pull back

$$\tau^{\mathbb{R}} = j_2^{\mathbb{R}*} \hat{\tau}^{\mathbb{R}} = j_2^{\mathbb{R}*} \hat{\varphi}^{\mathbb{R}} - \tilde{\varphi}^{\mathbb{R}} \quad (32)$$

is also.

The induced connection $\tilde{\varphi}^{\mathbb{R}}$ is torsion free, since $\tilde{\theta}^{\mathbb{R}}$, the pull back of the solder form, is again the solder form on $\text{Fr}_{\mathbf{SO}}(\mu^{-1}(0), N)$. We pull back the structure equation $d\hat{\theta}^{\mathbb{R}} + \hat{\varphi}^{\mathbb{R}} \wedge \hat{\theta}^{\mathbb{R}} = 0$ to get

$$d\tilde{\theta}^{\mathbb{R}} + (j_2^{\mathbb{R}*} \hat{\varphi}^{\mathbb{R}}) \wedge \tilde{\theta}^{\mathbb{R}} = d\tilde{\theta}^{\mathbb{R}} + (\tilde{\varphi}^{\mathbb{R}} + \tau^{\mathbb{R}}) \wedge \tilde{\theta}^{\mathbb{R}} = 0. \quad (33)$$

Since $\tilde{\theta}^{\mathbb{R}}$ has values in \mathbb{R}^{4n+k} , we can split the equation into the following two equations

$$d\tilde{\theta}^{\mathbb{R}} + \tilde{\varphi}^{\mathbb{R}} \wedge \tilde{\theta}^{\mathbb{R}} = 0, \quad (34)$$

$$\tau^{\mathbb{R}} \wedge \tilde{\theta}^{\mathbb{R}} = 0, \quad (35)$$

which shows that $\tilde{\varphi}^{\mathbb{R}}$ is indeed torsion free.

$\tau^{\mathbb{R}}$ splits naturally into two forms with values in the top right matrices and bottom left matrices. Let $\tau_1^{\mathbb{R}}$ denote the one that has values in the bottom left. Hence we have the splitting

$$j_2^{\mathbb{R}*} \hat{\varphi}^{\mathbb{R}} = \begin{pmatrix} \phi_1^{\mathbb{R}} & -(\tau_1^{\mathbb{R}})^t \\ \tau_1^{\mathbb{R}} & \phi_2^{\mathbb{R}} \end{pmatrix}. \quad (36)$$

Using lemma (2.10) to identify $\tau_1^{\mathbb{R}}$ with a $(2, 1)$ -tensor on $\mu^{-1}(0)$, via

$$s(\tau_1^{\mathbb{R}})(\xi, \eta) = p\tau_1^{\mathbb{R}}(\bar{\xi})\tilde{\theta}^{\mathbb{R}}(\bar{\eta}), \quad (37)$$

where p is a frame in $F(\mu^{-1}(0), M)$ and $\bar{\xi}, \bar{\eta}$ are lifts (compare lemma (2.10)).

Proposition 4.1 (Second fundamental form). *$s(\tau_1^{\mathbb{R}})$ is the second fundamental form of $\mu^{-1}(0)$ in M .*

Proof. In the next subsection we will show that $\tilde{\varphi}^{\mathbb{R}}$ is the pull back of the Levi Civita connection on $\text{Fr}_{\text{SO}}(\mu^{-1}(0))$. The covariant derivative of a connection φ with soldering form θ is given by

$$\nabla_t X = p(\bar{t}\theta(\bar{X}) + \varphi(\bar{t})\theta(\bar{X}_p)), \quad (38)$$

where \bar{t} and \bar{X} are lifts of the tangent vector t and vector field X to a frame p (see e.g. [2, 6.4], but note that this book has a very unusual sign convention for the second fundamental form). Hence the second fundamental form is given by

$$\text{II}(X, Y) = \nabla_X^M Y - \nabla_X^{\mu^{-1}(0)} Y = p(j_2^{\mathbb{R}*} \hat{\varphi}^{\mathbb{R}}(\bar{X}) - \phi_1^{\mathbb{R}}(\bar{X}))\tilde{\theta}^{\mathbb{R}}(\bar{Y}) \quad (39)$$

$$= p(j_2^{\mathbb{R}*} \hat{\varphi}^{\mathbb{R}}(\bar{X}) - \tilde{\varphi}^{\mathbb{R}}(\bar{X}))\tilde{\theta}^{\mathbb{R}}(\bar{Y}) = p\tau^{\mathbb{R}}(\bar{X}_p)\tilde{\theta}^{\mathbb{R}}(\bar{Y}_p). \quad (40)$$

Here we have used that X and Y are tangent to $\mu^{-1}(0)$ and hence $\phi_2^{\mathbb{R}}(\bar{X}_p)\tilde{\theta}^{\mathbb{R}}(\bar{Y}_p) = 0$. Note that II is symmetric, because $\tau^{\mathbb{R}} \wedge \tilde{\theta}^{\mathbb{R}} = 0$, by equation (35). Since the second fundamental form is only defined for tangent vectors to $\mu^{-1}(0)$ and takes values orthogonal to $\mu^{-1}(0)$, we have to restrict $\tau^{\mathbb{R}}$ to $\tau_1^{\mathbb{R}}$ as described above. \square

Proposition 4.2 (Second fundamental form as Hessian). *Let $f: M \rightarrow V$ be a smooth map, where M is a Riemannian manifold and V a vector space. Assume further, that $0 \in V$ is a regular value. $Df: TM \rightarrow V$ identifies every fiber of the bundle $Tf^{-1}(0)^\perp$ with V , and under this identification the negative of the Hessian matrix of f equals the second fundamental form of $f^{-1}(0)$ in M .*

Proof. The first claim is just the dimension formula for a linear map,

$$Df_p: Tf^{-1}(0) \oplus Tf^{-1}(0)^\perp \rightarrow V, \quad (41)$$

which has kernel $Tf^{-1}(0)$. Note that the second equality only holds for vector fields tangent to $f^{-1}(0)$, since the second fundamental form is only defined for these. Let X and Y be vector fields tangent to $f^{-1}(0)$. Then

$$\begin{aligned} \text{Hess}(f)(X, Y) &= X(Yf) - Df(\nabla_X^M Y) \\ &= X(\underbrace{Df(Y)}_{=0}) - \underbrace{Df(\nabla_X^{\mu^{-1}(0)} Y)}_{=0} - Df \Pi(X, Y) \\ &= -Df \Pi(X, Y) \end{aligned} \quad (42)$$

□

In this sense, $\tau_1^{\mathbb{R}}$ is associated with the $- \text{Hess}(\mu)$ by the two aforementioned propositions.

4.4 Forms on $\text{Fr}_{\text{SO}}(\mu^{-1}(0))$

Recall that the torsion free connection $\tilde{\varphi}^{\mathbb{R}}$ decomposes into two one forms $\phi_1^{\mathbb{R}}$ and $\phi_2^{\mathbb{R}}$. $\phi_1^{\mathbb{R}}$ with values in $\mathfrak{so}(4n+k)$ induces a connection on $\text{Fr}_{\text{SO}}(\mu^{-1}(0))$, because

$$\phi_1^{\mathbb{R}}((Dk_1^{\mathbb{R}})^{-1}(0)) = 0, \quad (43)$$

$$R_g^* \phi_1^{\mathbb{R}} = \phi_1^{\mathbb{R}} \quad \forall g \in O(3k) \subset O(4m), \quad (44)$$

which is true because $Dk_1^{\mathbb{R}}: \mathfrak{so}(4n+k) \oplus \mathfrak{so}(3k) \rightarrow \mathfrak{so}(4n+k)$ is the projection. It allows us to define

$$\varphi^{\mu^{-1}(0)}(\eta) = \phi_1^{\mathbb{R}}(\tilde{\eta}), \quad \tilde{\eta} \in (Dk_1^{\mathbb{R}})^{-1}_q(\eta), \quad (45)$$

i.e. $k_1^{\mathbb{R}*} \varphi^{\mu^{-1}(0)} = \phi_1^{\mathbb{R}}$. Since the solder form on $\mu^{-1}(0)$ pulled back to $F(\mu^{-1}(0), M)$ is the form $\tilde{\theta}^{\mathbb{R}}$, we get the equation

$$d\theta^{\mu^{-1}(0), \mathbb{R}} + \varphi^{\mu^{-1}(0)} \wedge \theta^{\mu^{-1}(0), \mathbb{R}} = 0, \quad (46)$$

and see that $\varphi^{\mu^{-1}(0)}$ is the unique Levi Civita connection on $\mu^{-1}(0)$.

4.5 Riemannian Submersions

The next step involves understanding Riemannian submersions on the level of frame bundles. Since there is no exposition of this known to the author, we will describe it in a general setting, and apply it to the reduction afterwards.

Let us at this point recall the basics of the Riemannian submersion theory of O'Neill [6]. A Riemannian submersion $\pi: M^m \rightarrow B^b$ is a smooth map between two Riemannian manifolds such that π is a submersion and $D\pi_x|_{H_x}: H_x \rightarrow T_{\pi(x)}B$ is a isometry for all $x \in M$, where H_x is the orthogonal complement of $\ker(D\pi) \subset T_x M$.

To such a Riemannian submersion we may associate two important $(2,1)$ -tensor fields on M ,

$$T_X Y = \mathcal{H}\nabla_{\mathcal{V}X}^M \mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{V}X}^M \mathcal{H}Y \quad (47)$$

$$A_X Y = \mathcal{H}\nabla_{\mathcal{H}X}^M \mathcal{V}Y + \mathcal{V}\nabla_{\mathcal{H}X}^M \mathcal{H}Y, \quad (48)$$

where \mathcal{H} and \mathcal{V} are the horizontal and vertical projection in TM , respectively. T is known to be the second fundamental form of each fiber (if vertical vector fields are plugged in), whereas A is related to the obstruction to integrability of the horizontal distribution on M . An important fact is that

$$A_X Y = \frac{1}{2} \mathcal{V}[X, Y], \quad (49)$$

for horizontal vector fields X and Y . If the Riemannian submersion $\pi: M \rightarrow B$ should also happen to be a principal bundle, and we fix the connection corresponding to the horizontal subspaces, then $2A_X Y = -R(X, Y)$, where $R(X, Y)$ is the curvature of the connection, if we identify the vertical tangent space with the Lie algebra as usual.

In the world of principal bundles this can be expressed the following way. Let $\text{Fr}(M)$ be the principal bundle of frames and $\text{Fr}(B, M)$ the reduction to adapted frames on M . Here a frame is adapted if it respects the splitting of TM into horizontal and vertical parts, i.e.

$$\text{Fr}(B, M) = \{p \in \text{Fr}(M) : \text{im}(p|_{\mathbb{R}^b}) \text{ is horizontal}\}. \quad (50)$$

Then a pull back of the Levi Civita connection ϕ on $\text{Fr}(M)$ and the solder form θ gives, after a suitable projection, a connection ψ on $\text{Fr}(B, M)$ with structure equation

$$d\theta' + \psi \wedge \theta' + \tau \wedge \theta' = 0, \quad (51)$$

where θ' is the pull back of the solder form, ψ the projected connection and $\tau = i^*\phi - \psi$, where $i: \text{Fr}(B, M) \rightarrow \text{Fr}(M)$ is the inclusion. We see that τ is an obstruction to the integrability of the horizontal distribution, because for a product manifold $M = M_1 \times M_2$ we have the commutative diagram

$$\begin{array}{ccccc}
 & & \text{Fr}(M) & & \\
 & & \uparrow & & \\
 \text{Fr}(M_1) & \longleftarrow & \text{Fr}(M_1, M_2) & \longrightarrow & \text{Fr}(M_2) \\
 \downarrow & & \downarrow & & \downarrow \\
 M_1 & \longleftarrow & M & \longrightarrow & M_2
 \end{array}$$

and the connection on $\text{Fr}(M)$ reduces to a connection on $\text{Fr}(M_1, M_2)$, which is the sum of the connections pulled back from $\text{Fr}(M_i)$. On the other hand, from the construction of the last chapter, we also know that τ is related to the second fundamental forms of the fibers.

The notion of horizontal and vertical projection extends to horizontal forms on $\text{Fr}(B, M)$, via

$$\tau_h(\xi) = \tau(\overline{\mathcal{H}D\pi'(\xi)}) \quad (52)$$

$$\tau_v(\xi) = \tau(\overline{\mathcal{V}D\pi'(\xi)}), \quad (53)$$

where π' is the principal bundle map of $\text{Fr}(B, M)$ and the over line is a lift with respect to that map. It is easy to see that this is well defined for a horizontal form, since it does not depend on the choice of lift. Note also that by definition $\tau = \tau_h + \tau_v$. The following proposition is the main result of this section.

Proposition 4.3 (O'Neill on Principal Bundles). τ_v corresponds to T and τ_h corresponds to A .

Proof. Note that τ is described by the difference of the connection on $\text{Fr}(M)$ and the connection on $\text{Fr}(B, M)$. The connection on $\text{Fr}(M)$ gives rise to the covariant derivative ∇^M , and the connection on $\text{Fr}(B, M)$ to $\tilde{\nabla}$. As we have shown before, the connection extended from $\tilde{\nabla}$ splits into two connections which are the Levi Civita connection on the fibers and the horizontal submanifolds, if they exist. Even if they do not, a quick inspection of equation (38), using the matrix form of the reduced connection, shows that

$$\tilde{\nabla}_\xi X = \mathcal{H}\nabla_\xi^M X \quad (54)$$

if ξ and X are horizontal and

$$\tilde{\nabla}_\eta Y = \mathcal{V}\nabla_\eta^M Y, \quad (55)$$

if η and Y are vertical. The unique extension of this to $\text{Fr}(M)$ gives the connection

$$\hat{\nabla}_\chi Z := \mathcal{H}\nabla_\chi^M \mathcal{H}Z + \mathcal{V}\nabla_\chi^M \mathcal{V}Z, \quad (56)$$

for χ an arbitrary tangent vector and Z an arbitrary vector field on M . This can be verified by showing that the above is indeed a covariant derivative on M and that it restricts to $\tilde{\nabla}$ if both χ and Z are vertical, or both are horizontal. The latter is immediately clear, the former some simple calculations.

We see now, that

$$\begin{aligned} \nabla_\chi^M Z &= \mathcal{H}\nabla_{\mathcal{H}\chi}^M \mathcal{H}Z + \mathcal{H}\nabla_{\mathcal{V}\chi}^M \mathcal{H}Z + \mathcal{H}\nabla_{\mathcal{H}\chi}^M \mathcal{V}Z + \mathcal{H}\nabla_{\mathcal{V}\chi}^M \mathcal{V}Z \\ &\quad + \mathcal{V}\nabla_{\mathcal{H}\chi}^M \mathcal{H}Z + \mathcal{V}\nabla_{\mathcal{V}\chi}^M \mathcal{H}Z + \mathcal{V}\nabla_{\mathcal{H}\chi}^M \mathcal{V}Z + \mathcal{V}\nabla_{\mathcal{V}\chi}^M \mathcal{V}Z \\ &= A_\chi Z + T_\chi Z + \hat{\nabla}_\chi Z, \end{aligned} \quad (57)$$

hence the difference of connections indeed gives $A + T$. Finally, notice that if χ is horizontal then T vanishes, as does τ_v . If on the other hand χ is vertical, then A vanishes, as does τ_h . \square

The principal bundle of frames $\text{Fr}(B)$ of B can be pulled back to M via π . The Levi Civita connection ϕ^B on $\text{Fr}(B)$ can also be pulled back to a connection $\tilde{\phi}$ on $\pi^* \text{Fr}(B)$ together with the structure equation

$$\tilde{\phi} + \tilde{\theta}_B \wedge \tilde{\phi} = 0, \quad (58)$$

where $\tilde{\theta}_B$ is the pull back of the solder form θ_B on $\text{Fr}(B)$. If we pull this solder form into $\text{Fr}(B, M)$, we get a form θ'_B , where the obvious restriction map is used $k: \text{Fr}(B, M) \rightarrow \pi^* \text{Fr}(B)$. A calculation similar to that in remark (2.9) shows that θ'_B agrees with the part of θ' , that has values in \mathbb{R}^b . If we split θ' into two parts, θ_1 and θ_2 with values in \mathbb{R}^b and \mathbb{R}^{m-b} , and ψ into ψ_1 and ψ_2 with values in $\mathfrak{so}(b)$ and $\mathfrak{so}(m-b)$, then the structural equation (51) of ψ decomposes into

$$d\theta_1 + \psi_1 \wedge \theta_1 + \tau \wedge \theta_2 = 0 \quad (59)$$

$$d\theta_2 + \psi_2 \wedge \theta_2 + \tau \wedge \theta_1 = 0. \quad (60)$$

If we restrict the first equation to π -horizontal vectors, the last term vanishes and we see that ψ_1 is the Levi Civita connection pulled back from B . Such a restriction also turns τ into τ_h and we get the formula

$$k^* \pi^* \phi^B + \tau_h = i^* \phi_M, \quad (61)$$

on $\text{Fr}(B, M)$, if we restrict to vectors lifted from B . This is the recovery of O'Neill's formula for the connections [6, Lemma 3.4].

4.6 Forms on $\text{Fr}_{\text{SO}}(N, \mu^{-1}(0))$

Applying the last section to the reduction $\text{Fr}_{\text{SO}}(N, \mu^{-1}(0))$ of $\text{Fr}_{\text{SO}}(\mu^{-1}(0))$ on $\mu^{-1}(0)$, we get the equation

$$j_3^{\mathbb{R}*} \varphi^{\mu^{-1}(0)} = \psi_1 + \psi_2 + \tau', \quad (62)$$

where ψ_1 is the pull back of the Levi Civita connection on N .

4.7 Forms on $\text{Fr}_{\text{Sp}}(N, M)$

Now we will do a similar construction on the quaternionic side of the reduction for $\text{Fr}_{\text{Sp}}(N, M)$. As with $\text{Fr}_{\text{SO}}(\mu^{-1}(0), M)$, $\text{Fr}_{\text{Sp}}(N, M)$ will in general not be horizontal in $\iota^* \text{Fr}_{\text{Sp}}(M)$. Using Proposition 2.7, we construct a connection $\tilde{\varphi}^{\mathbb{H}}$ with the decomposition

$$\mathfrak{sp}(\mathfrak{m}) = \mathfrak{sp}(\mathfrak{n}) \oplus \mathfrak{o}(\mathfrak{k}) \oplus \mathfrak{f}, \quad (63)$$

induced by an inclusion of $\mathbf{Sp}(\mathfrak{n}) \times \mathbf{SO}(\mathfrak{k})$ in $\mathbf{Sp}(\mathfrak{m})$ as described in the beginning. As before, the obvious choice of complement will satisfy the necessary condition (7).

We get the projected connection form $\tilde{\varphi}^{\mathbb{H}}$ which decomposes into two equivariant one-forms $\phi_1^{\mathbb{H}}$ and $\phi_2^{\mathbb{H}}$ with values in $\mathfrak{sp}(\mathfrak{n})$ and $\mathfrak{so}(\mathfrak{k})$ respectively and a difference form $\tau^{\mathbb{H}}$ with

$$\phi_1^{\mathbb{H}} + \phi_2^{\mathbb{H}} + \tau^{\mathbb{H}} = j_2^{\mathbb{H}*} \hat{\varphi}^{\mathbb{H}}. \quad (64)$$

5 Final Result

5.1 Preparation

Let us recall the connections of the real reductions. On $\text{Fr}_{\text{SO}}(\mu^{-1}(0), M)$ we have equation (32)

$$\phi_1^{\mathbb{R}} + \phi_2^{\mathbb{R}} + \tau^{\mathbb{R}} = j_2^{\mathbb{R}*} \hat{\varphi}^{\mathbb{R}}, \quad (65)$$

where $\hat{\phi}^{\mathbb{R}}$ is the pull back of the Levi Civita connection on M . $\phi_1^{\mathbb{R}}$ is the pull back of the Levi-Civita connection of $\mu^{-1}(0)$, which in turn decomposes on $\text{Fr}_{\text{SO}}(N, \mu^{-1}(0))$ according to equation (62).

The connection $\psi_1 + \psi_2$ on $\text{Fr}_{\text{SO}}(N, \mu^{-1}(0))$ can be extended back to a connection $\tilde{\psi}_1 + \tilde{\psi}_2$ on $\text{Fr}_{\text{SO}}(\mu^{-1}(0))$, so that we have

$$\tilde{\psi}_1 + \tilde{\psi}_2 + \tilde{\tau}' = \varphi^{\mu^{-1}(0)}, \quad (66)$$

where $\tilde{\tau}'$ is defined by this equation (and hence the pull back of it is τ' .) So if we pull back this equation to $\text{Fr}_{\text{SO}}(\mu^{-1}(0), M)$, we get

$$k_1^{\mathbb{R}*} \tilde{\psi}_1 + k_1^{\mathbb{R}*} \tilde{\psi}_2 + k_1^{\mathbb{R}*} \tilde{\tau}' = \phi_1^{\mathbb{R}}, \quad (67)$$

and combining this with (65)

$$k_1^{\mathbb{R}*} \tilde{\psi}_1 + k_1^{\mathbb{R}*} \tilde{\psi}_2 + k_1^{\mathbb{R}*} \tilde{\tau}' + \phi_2^{\mathbb{R}} + \tau^{\mathbb{R}} = j_2^{\mathbb{R}*} \hat{\phi}^{\mathbb{R}}. \quad (68)$$

Since $i_3^* \hat{\phi}^{\mathbb{R}} = \hat{\phi}^{\mathbb{H}}$, we can identify the right hand side of the equation above and of (64) if we pull back by i_3 ,

$$i_3^* \left(k_1^{\mathbb{R}*} \tilde{\psi}_1 + k_1^{\mathbb{R}*} \tilde{\psi}_2 + k_1^{\mathbb{R}*} \tilde{\tau}' + \phi_2^{\mathbb{R}} + \tau^{\mathbb{R}} \right) = \phi_1^{\mathbb{H}} + \phi_2^{\mathbb{H}} + \tau^{\mathbb{H}}. \quad (69)$$

To understand which terms correspond, it is a good idea to visualize where the different forms take their values. If we identify \mathbb{H}^n with \mathbb{R}^{4n} such that $a + ib + jc + kd$ gets mapped to (a, b, c, d) ($a, b, c, d \in \mathbb{R}^n$), we identify $n \times n$ quaternionic matrices $A + iB + jC + kD$ with $4n \times 4n$ real matrices of the form

$$\begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix}. \quad (70)$$

If we use a frame $p \in \text{Fr}_{\text{Sp}}(N, M)$ to identify $\iota^*(TM)$ with \mathbb{R}^{4m} , we see that both sides of the equations take values in matrices of the form

$$\begin{pmatrix} M_1 & -M_2^t \\ M_2 & M_3 \end{pmatrix}, \quad (71)$$

where M_1 is a $4n \times 4n$, M_2 a $4k \times 4n$ and M_3 a $4k \times 4k$ block matrix of the type given above. Using the quaternionic splitting, we can decompose the M_i

into A_i, B_i, C_i and D_i . Note that in M_3 only A_3 (the diagonal) is non vanishing, because of the inclusion $SO(k) \hookrightarrow Sp(k)$, $A \mapsto A + iA + jA + kA$.

The components of the matrices M_i are of course only defined up to the choice of frame $p \in Fr_{Sp}(N, M)$. However, two different frames differ by a matrix in $Sp(n) \times SO(k)$, which leaves the components of M_3 and the component-rows of M_2 invariant. M_1 and the columns of M_2 get transformed by conjugation with a \mathbb{H} -linear matrix.

Define the matrix M_2^1 to be the first k rows of M_2 , M_2^2 to be the other $3k$ rows and M_3^1 as A_3 , $(M_3^2)^t = (B_3, C_3, D_3)$ and M_3^3 as the matrix M_3 without the first k columns and first k rows. Hence we may write (71) as

$$\begin{pmatrix} M_1 & -(M_2^1)^t & -(M_2^2)^t \\ M_2^1 & M_3^1 & -(M_3^2)^t \\ M_2^2 & M_3^2 & M_3^3 \end{pmatrix}.$$

Starting with the right hand side of the equation (69), $\phi_1^{\mathbb{H}}$ takes values M_1 , $\phi_2^{\mathbb{H}}$ in M_3 and $\tau^{\mathbb{H}}$ the remaining M_2 matrix. On the left hand side, $\tilde{\psi}_1$ takes values in the M_1 , $\tilde{\psi}_2$ in M_3^1 , $\tilde{\tau}'$ in M_2^1 , $\phi_2^{\mathbb{R}}$ in M_3^3 and $\tau^{\mathbb{R}}$ in the remaining M_2^2 and M_3^2 matrices.

5.2 The Results

The equations (69) and the following analysis of the previous section allows us to recover some of the results from [3]. First we see that

$$i_3^* k_1^{\mathbb{R}*} \tilde{\psi}_1 = \phi_1^{\mathbb{H}} \quad \Rightarrow \quad k^* \psi_1 = \phi_1^{\mathbb{H}}, \quad (72)$$

because both sides take values in M_1 . If we pull back the Levi-Civita connection on $Fr_{SO}(N)$ to $Fr_{Sp}(N, M)$ via $Fr_{Sp}(N)$, we get $\phi_1^{\mathbb{H}}$ because of this equation. Hence the pull back to $Fr_{Sp}(N)$ takes values in \mathbb{H} -linear matrices, in other words the connection reduces to one on $Fr_{Sp}(N)$. This shows that N is indeed a hyperkähler manifold.

A more constructive argument can be given by noting that the Levi-Civita connection on M is G -invariant, for the canonical choice of extension of the G action to $Fr_{SO}(M)$. This remains true for $\phi_1^{\mathbb{H}}$ and a careful examination shows that it can be pushed down to $Fr_{Sp}(N)$.

If we continue with M_3 , we see that for $\xi \in \mathfrak{g}$, $\Pi(\cdot, \xi)$, which is described by $M_3^2 = 0$, vanishes.

The fact that M_3 is only non-vanishing on the diagonal, gives a connection between the covariant derivative on the fibers of $\pi: \mu^{-1}(0) \rightarrow N$, and the normal derivative of $\mu^{-1}(0)$ described by ϕ_2^R , i.e. $D_\xi Y := \text{pr}_{T\mu^{-1}(0)^\perp} \nabla_\xi^M Y$, for $\xi \in T\mu^{-1}(0)$ and $Y \in \Gamma(\mu^{-1}(0), T\mu^{-1}(0)^\perp)$ (see e.g. [5, VII]). Precisely, we have for all $A \in \{I, J, K\}$

$$\nabla_\xi^F X = d\mu^A \circ D_\xi(AK^\eta), \quad \forall \xi, \eta \in \mathfrak{g}, \quad (73)$$

where ∇^F is the connection on the fiber.

Let us now focus on M_2 . From proposition (4.1) we know that M_2^2 and M_3^2 give the second fundamental form and from proposition (4.3) we know that M_2^1 is $A + T$, the O'Neill tensors. Hence

$$M_2(\xi) = p^{-1} \circ \begin{pmatrix} (A_\xi + T_\xi)(\cdot) & (A_\xi + T_\xi)(I\cdot) & (A_\xi + T_\xi)(J\cdot) & (A_\xi + T_\xi)(K\cdot) \\ \Pi^I(\xi, \cdot) & \Pi^I(\xi, I\cdot) & \Pi^I(\xi, J\cdot) & \Pi^I(\xi, K\cdot) \\ \Pi^J(\xi, \cdot) & \Pi^J(\xi, I\cdot) & \Pi^J(\xi, J\cdot) & \Pi^J(\xi, K\cdot) \\ \Pi^K(\xi, \cdot) & \Pi^K(\xi, I\cdot) & \Pi^K(\xi, J\cdot) & \Pi^K(\xi, K\cdot) \end{pmatrix} \circ p, \quad (74)$$

where Π^A is the second fundamental form of $\mu^{-1}(0) \hookrightarrow M$ projected onto $A\mathfrak{g} \subset T\mu^{-1}(0)^\perp$ and $p \in \text{Fr}_{\mathbf{Sp}}(N, M)$ is a frame (restricted in a suitable way). Using the form (70) of the matrix, we get the following results (recall the notation $\iota^*(TM) = H \oplus \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{H}$).

If $\xi \in H$ and $\cdot \in H$, then the first row of M_2 becomes $-\frac{1}{2}R(\xi, \cdot), \dots$, where R is the curvature of $\mu^{-1}(0) \rightarrow N$ as discussed before. This yields that for all $\xi, \eta \in H$,

$$-\frac{1}{2}R(\xi, \eta) = \Pi^I(\xi, I\eta) = \Pi^J(\xi, J\eta) = \Pi^K(\xi, K\eta). \quad (75)$$

Here $\Pi^I = d\mu^I \circ \Pi$. Note that this in particular implies that R is hyperholomorphic, i.e. of type $(1, 1)$ with respect to all complex structures (on N , viewing R as a two form on N).

If $\xi \in \mathfrak{g}$ and $\cdot \in H$, then the first row becomes $T_\xi \cdot = \mathcal{V} \nabla_\xi^{\mu^{-1}(0)} \cdot, \dots$, where $\nabla^{\mu^{-1}(0)}$ is the Levi-Civita connection on $\mu^{-1}(0)$ and \mathcal{V} is the vertical projection in $T\mu^{-1}(0)$ from $\pi: \mu^{-1}(0) \rightarrow N$. This can be described as the negative of the Weingarten map $\mathcal{W}_\xi(\cdot)$ of the fibers of π . Hence we get for all $\xi \in \mathfrak{g}, \eta \in H$,

$$-\mathcal{W}_\xi(\eta) = \Pi^I(\xi, I\eta) = \Pi^J(\xi, J\eta) = \Pi^K(\xi, K\eta).$$

However, since Π is symmetric, $\Pi(\xi, \cdot) = 0$, hence the Weingarten map of the fibers vanish, in other words, the fibers are totally geodesic.

If $\xi \in \mathcal{H}$ and $\cdot \in \mathfrak{g}$, the discussion needs to be carried out in $-M_2^t$. Using the formula for A and T (and that $\Pi(\xi, \cdot) = 0$), we see that

$$\text{pr}_{\mathcal{H}} \circ \nabla_{\xi}^{\mu^{-1}(0)} X = 0, \quad (76)$$

for all $\xi \in \mathcal{H}$ and $X \in \Gamma(\mu^{-1}(0), \mathfrak{g})$, which is already clear from $\mathcal{W}_X(\xi) = 0$. Both ξ and \cdot in \mathfrak{g} again yield that the second fundamental forms of the fibers of π vanish.

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